Lecture 11:
Basic Concepts of Wavefront Reconstruction

Astro 289

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Based on slides by Claire Max, Marcos van Dam, Lisa Poyneer the CfAO Summer School, and the hcipy PYWFS tutorial

What is wavefront reconstruction?

- Most wavefront sensors do not measure the phase directly, but instead measure the average derivative
- Most wavefront correctors are used to conjugate that phase on the mirror's surface
- We must reconstruct the phase from the WFS slopes, achieving the most accurate, lowest noise estimate possible in the least amount of computation

Method 1: Zonal Matrix Reconstruction

- The slope vector s contains x and y slopes for all valid subapertures in the pupil
- The phase ϕ contains all controllable actuators
- The basis set for reconstruction is the actuators
- We model the WFS measurement process as:

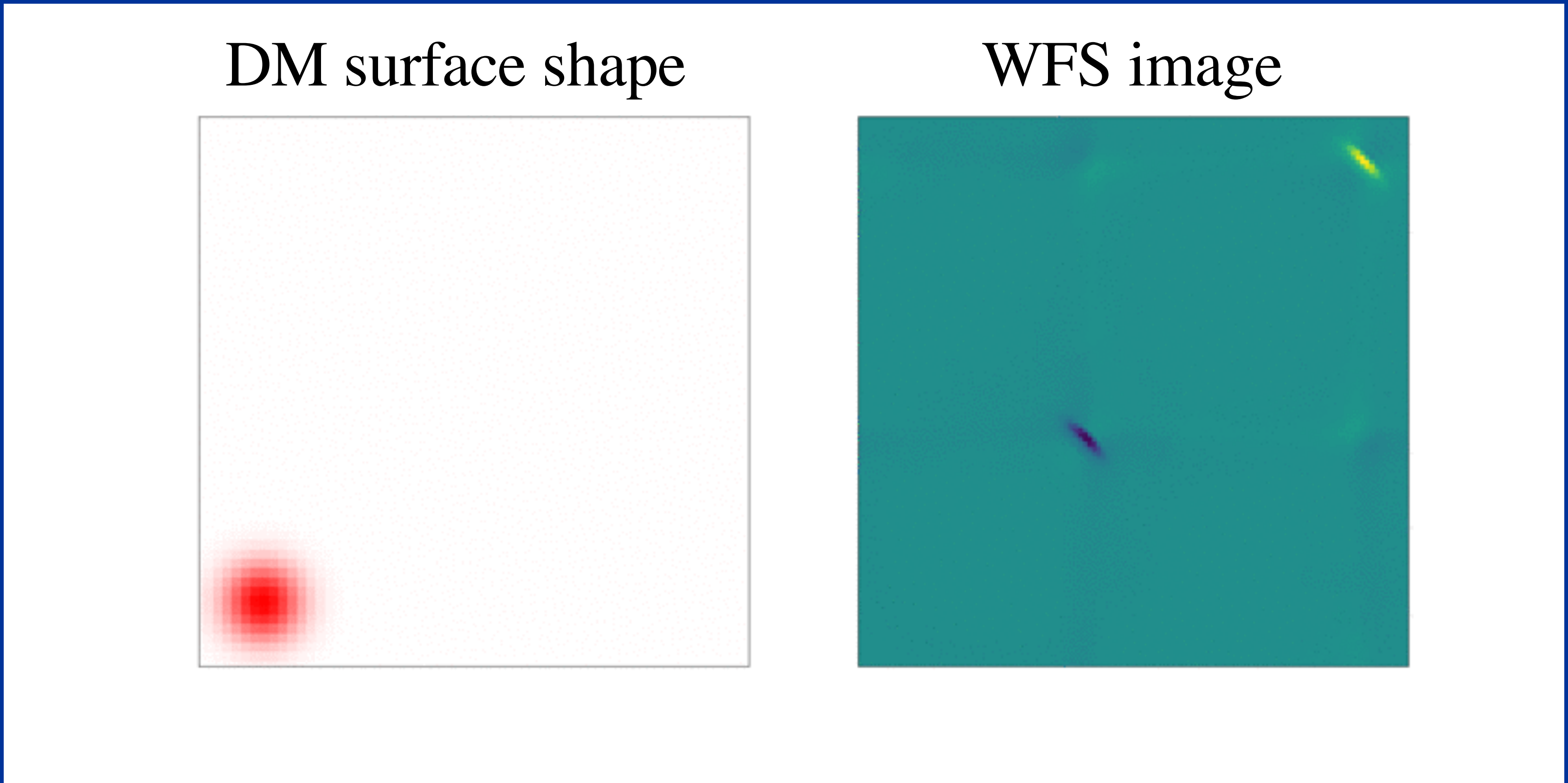
$$s = W\phi$$

- We can find the measurement matrix W experimentally

Experimental system matrix

- Poke one actuator at a time in the positive and negative directions and record the WFS centroids
- Set WFS centroid values from subapertures far away from the actuators to 0

Experimental system matrix



- We're left with a # pokes by # pixels matrix, which we're calling W

Method 1: Zonal Matrix Reconstruction

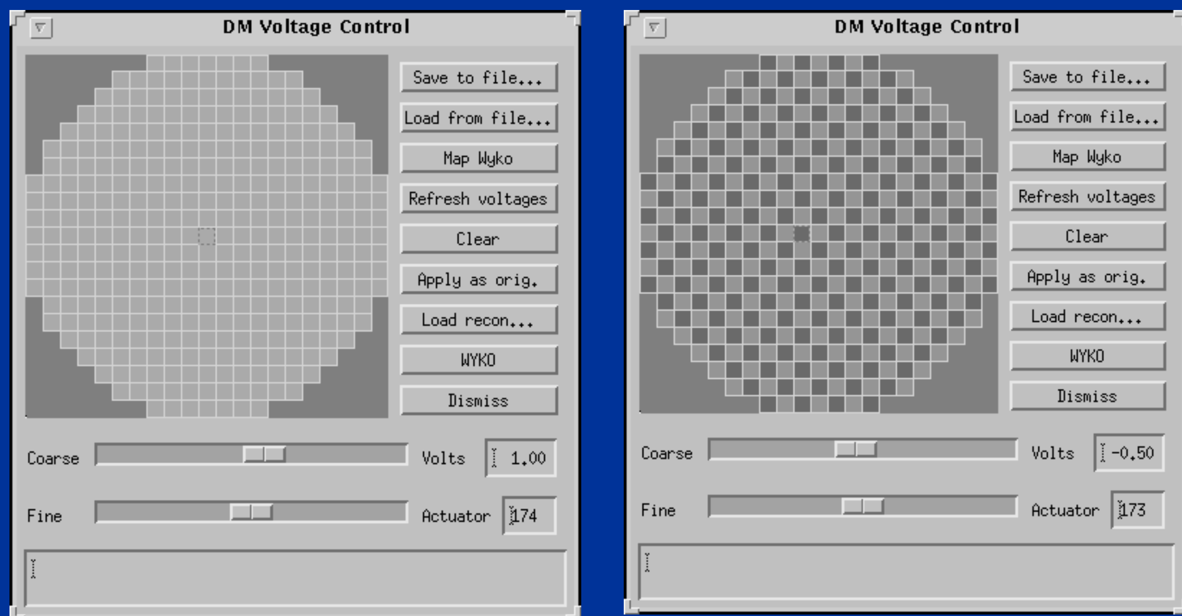
- Given the matrix W, we can convert phases to slopes via s = W ϕ
- In reality, we'll be measuring slopes with our WFS, and we'll want to use those slopes to estimate the phase via $\phi = W^{-1} s$
- W isn't square, so we'll compute the inverse using LS:

$$s = W\phi$$
$$W^T s = W^T W \phi$$
$$(W^T W)^{-1} W^T s = \phi$$

- Unfortunately, W^TW isn't invertible either -- it doesn't have full rank because the WFS is blind to some modes

Method 1: Zonal Matrix Reconstruction

- Two invisible modes are piston and waffle:



- Instead we'll compute the pseudo-inverse of W using singular value decomposition (SVD), giving us an expression for $\phi = W^{\dagger} s$

Singular-Value Decomposition (SVD)

- **If A is a square matrix, we can write:**

$$Av = \lambda v$$

- **Where v is a non-zero vector (the eigenvector) and λ is a scalar (the eigenvalue)**
- **If A is not square, we can equivalently compute the eigenvectors of A^TA. These are called the singular vectors of A**
- **The square roots of the eigenvalues of A^TA are called the singular values of A**

Singular-Value Decomposition (SVD)

- If W is a real m x n matrix, we can represent it as:

$$W = U\Sigma V^T$$

- U is an m x m matrix while columns are the eigenvectors of WW^T
- Σ is an m x n diagonal matrix whose diagonal values are the singular values of W
- V^T is the transpose of an n x n matrix whose columns are the eigenvectors of W^TW

Calculating the Pseudoinverse with SVD

- **Given $W = U\Sigma V^T$, the pseudo inverse of W is:**

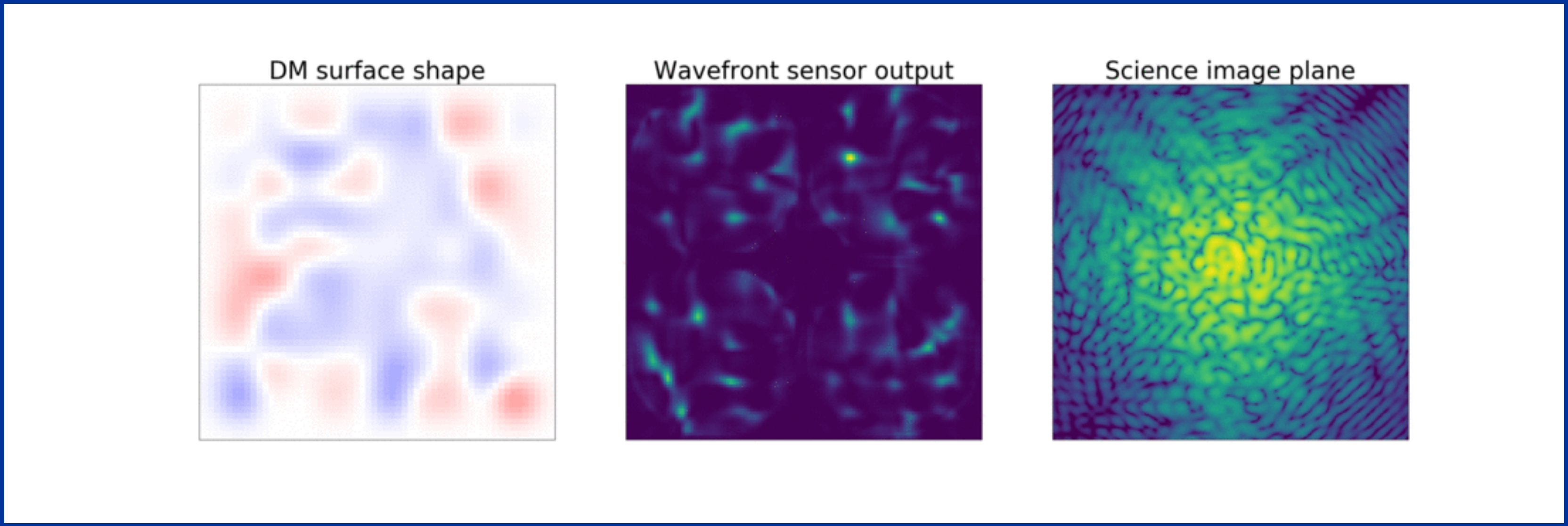
$$W^\dagger = V\Sigma^\dagger U^T$$

- **Where † is the pseudoinverse**
- **The pseudoinverse of Σ is just a diagonal matrix whose diagonal elements are the reciprocal of Σ's diagonal elements**

Method 1: Zonal Matrix Reconstruction

- **With the reconstruction matrix we found previously, we can reconstruct an example wavefront!**

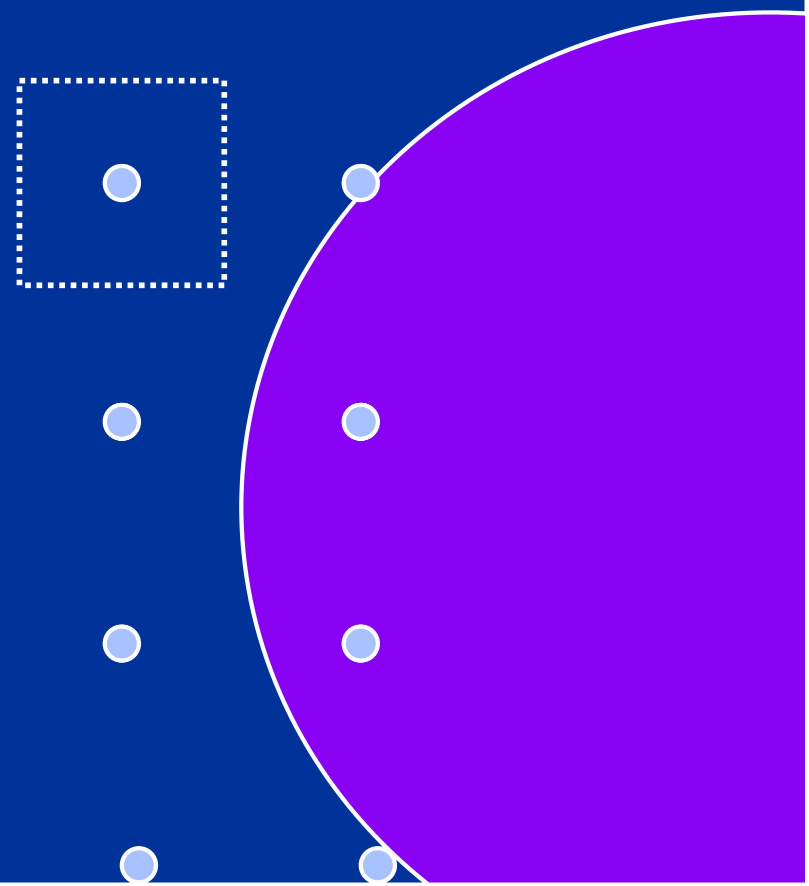
$$DM\_acts_t = (1\text{-leakage})\ DM\_acts_{t-1} - \text{gain}(W^{\dagger} \cdot \text{WFS image}_{t-1})$$



In this "leaky integrator" example, leakage = 0.01 and gain = 0.5

“Slaved” actuators

- Some actuators are located outside the pupil and do not directly affect the wavefront
- They are often “slaved” to the average value of its neighbors

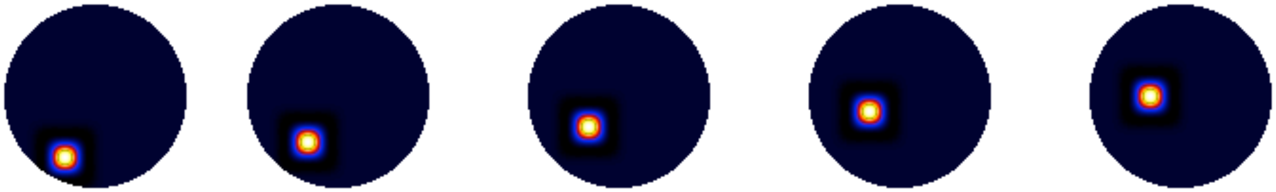


Method 2: Modal Matrix Reconstruction

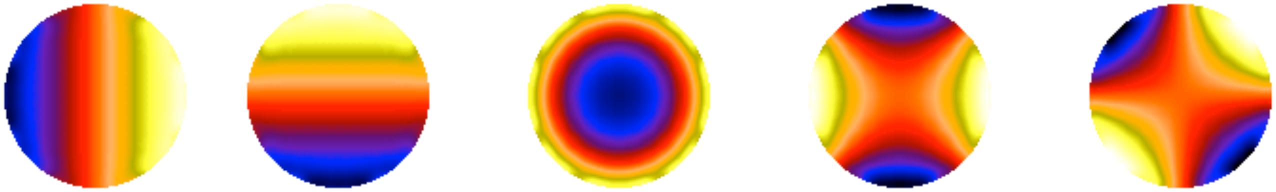
- We define an orthogonal modal basis set to represent the actuators: $\langle m_k, m_l \rangle = 0$ for $k \neq l$

Examples of modal basis sets

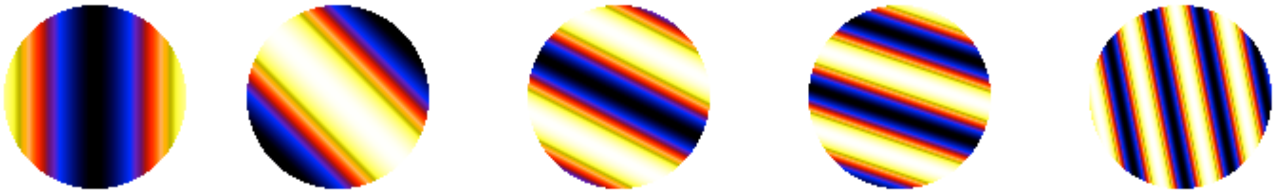
- Actuators



- Zernike modes



- Fourier modes



Method 2: Modal Matrix Reconstruction

- We define an orthogonal modal basis set to represent the actuators: $\langle m_k, m_l \rangle = 0$ for $k \neq l$
- We can analyze the phase in terms of modal coefficients with the inner product $c_k = \langle \phi, m_k \rangle$
- The phase is synthesized from the modal coefficients as

$$\phi = \sum_{k=0}^{n-1} c_k \frac{m_k}{\langle m_k, m_k \rangle}$$

- We can put the modes into rows or columns to produce the modal synthesis matrix *M*

Method 2: Modal Matrix Reconstruction

- Now, the slope measurement process is:

$$s = WM^{-1}c$$

- And the modal reconstruction is:

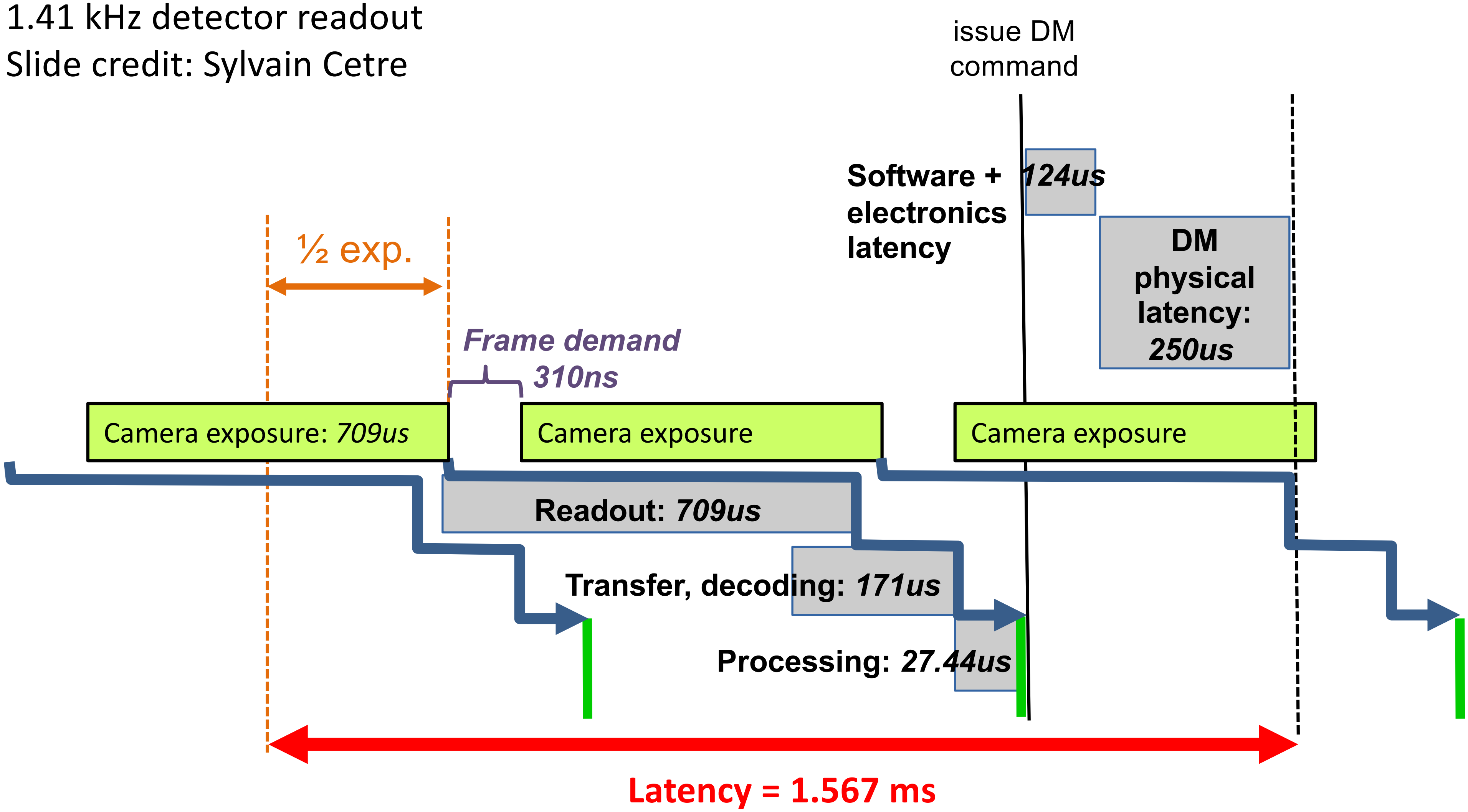
$$c = MW^{-1}s$$

- An advantage of modes is that they can be weighted and manipulated. E.g. we can easily remove piston, tip, and tilt from Zernike modes by zeroing the correct coefficients using matrix G:

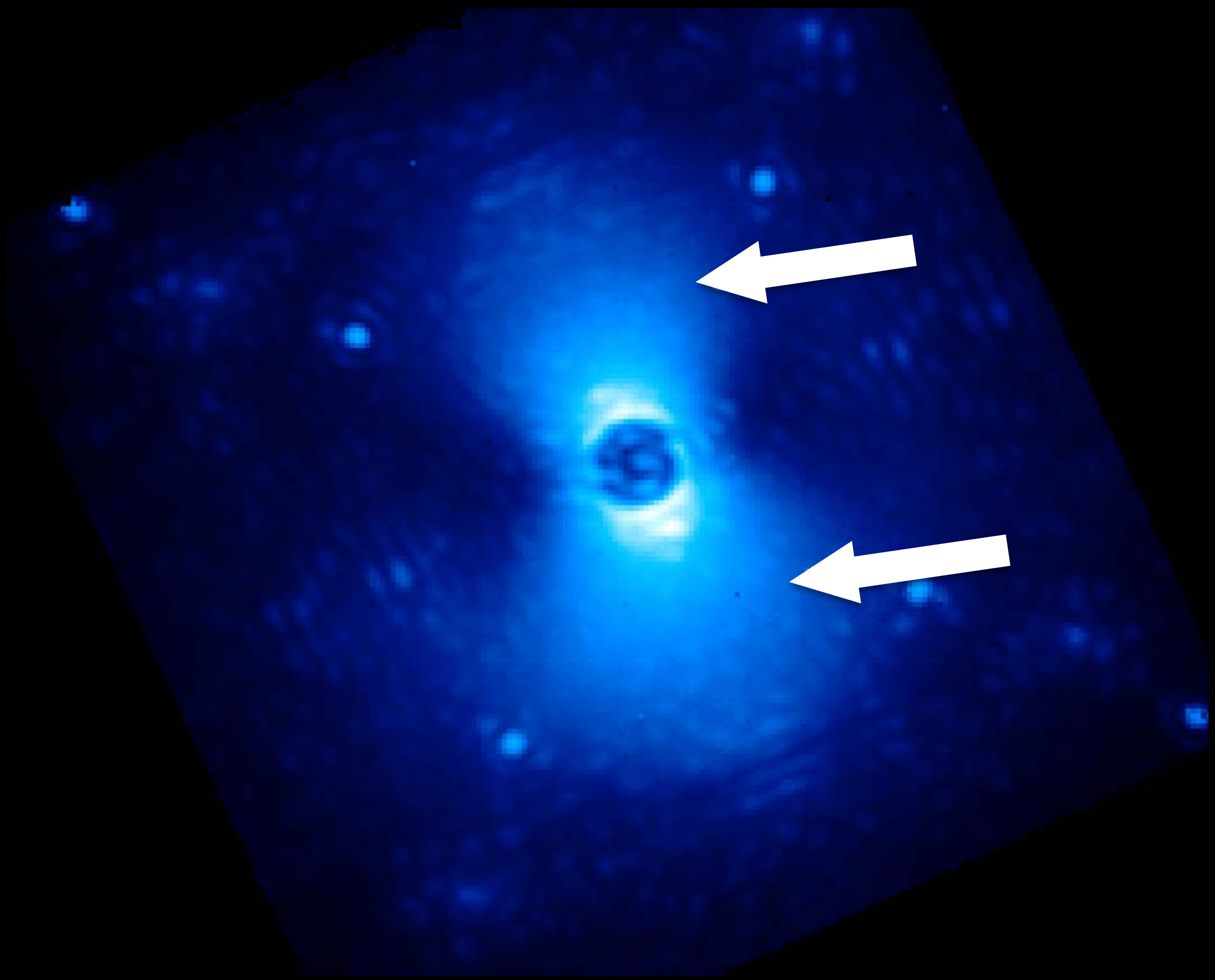
$$\phi = M^{-1}GMW^{-1}s$$

WF reconstruction & control don't happen instantaneously!

Keck II PYWFS Latency
1.41 kHz detector readout
Slide credit: Sylvain Cetre

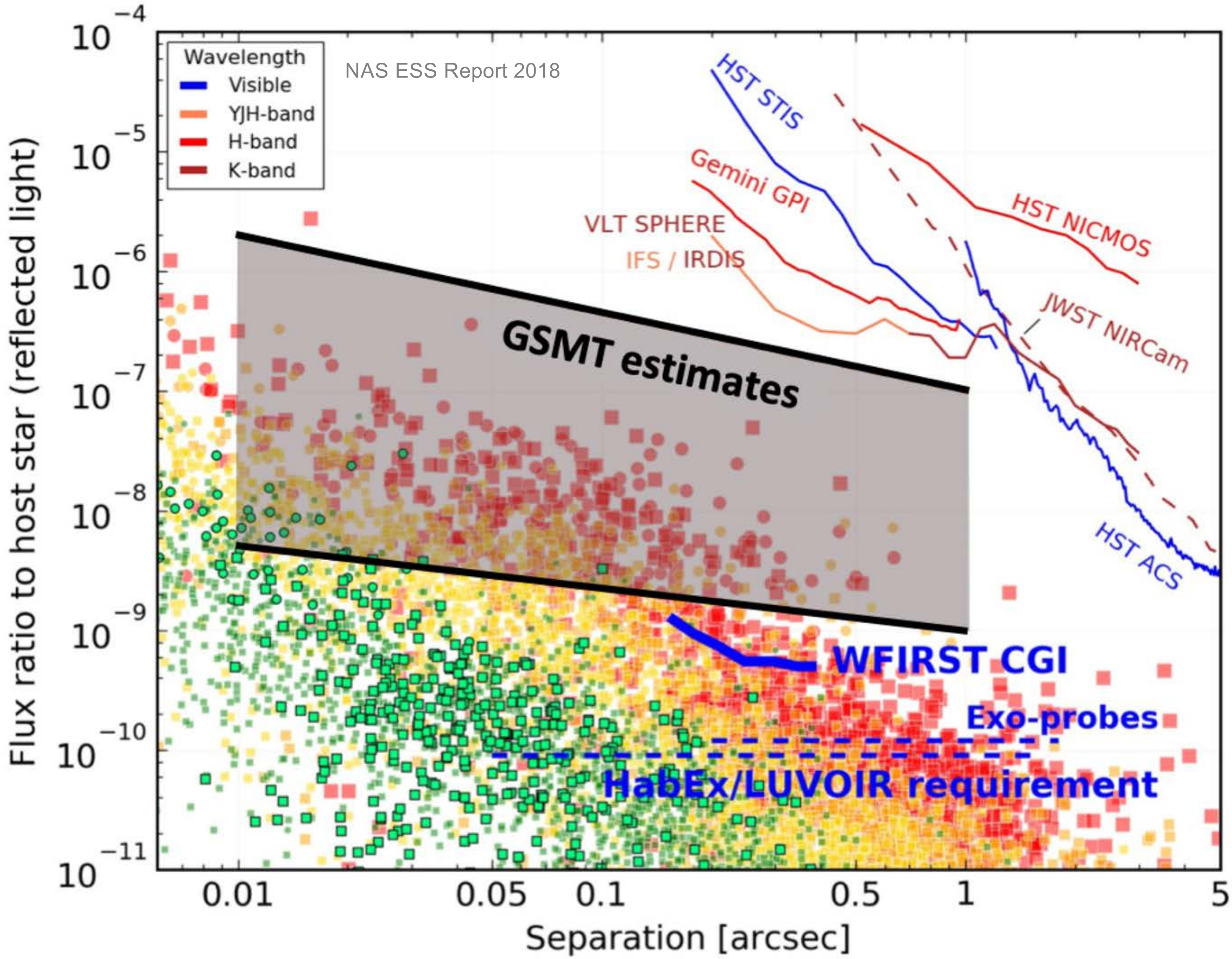


When the atmospheric coherence time is short compared with the AO system's speed, we start to see effects like the "butterfly"



Perrin/Maire/GPI Data Analysis Team

See also Madurowicz+18,19


Wavelength
Visible
YJH-band
H-band
K-band
NAS ESS Report 2018
HST STIS
Gemini GPI
VLT SPHERE
IFS / IRDIS
HST NICMOS
JWST NIRCam
HST ACS
GSMT estimates
WFIRST CGI
Exo-probes
HabEx/LUVOIR requirement
Flux ratio to host star (reflected light)
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
0.01
0.05
0.1
0.5
1
5
Separation [arcsec]

Predictive Wavefront Control

- Several methods for predictive control have been proposed since the 1990s, e.g. Fourier predictive control (Poyneer et al. 2007, 2008) and Empirical Orthogonal Functions (Guyon & Males 2017)
- Each method has a different way of computing the WF one time delay into the future based on past WF information
- Each method makes different assumptions about the data and the atmosphere

Guyon & Males 2017: put the *n* most recent wavefronts into a “history” vector h(t)

$$h(t) = \begin{bmatrix} w_0(t) \\ w_1(t) \\ \vdots \\ w_{m-1}(t) \\ \\ w_0(t-dt) \\ \vdots \\ w_{m-1}(t-dt) \\ \vdots \\ w_{m-1}(t-(n-1)dt) \end{bmatrix}$$

Guyon & Males 2017: find a filter **F** that minimizes the distance between **F** h(t) and the future wavefront ω(t+δt)

$$min_{\boldsymbol{F}^i} < ||\boldsymbol{F}^i \boldsymbol{h}(t) - \boldsymbol{\omega}_i(t + \delta t)||^2 >_t$$

Guyon & Males 2017: to compute the filter **F**, we'll use a "training set" of previous history vectors, **D**, and interpolate to find the corresponding training set **P** shifted one time delay later

Matrix of History Vectors:

$$\boldsymbol{D = [h(t)h(t-dt)...h(t-(l-1)dt]}$$

Matrix of "Future" Vectors: δt is the time lag

$$P = \begin{bmatrix} w_0(t+\delta t) & \dots & w_0(t-(l-1)dt+\delta t) \\ \vdots & \ddots & \vdots \\ w_{m-1}(t+\delta t) & \dots & w_{m-1}(t-(l-1)dt+\delta t) \end{bmatrix}$$

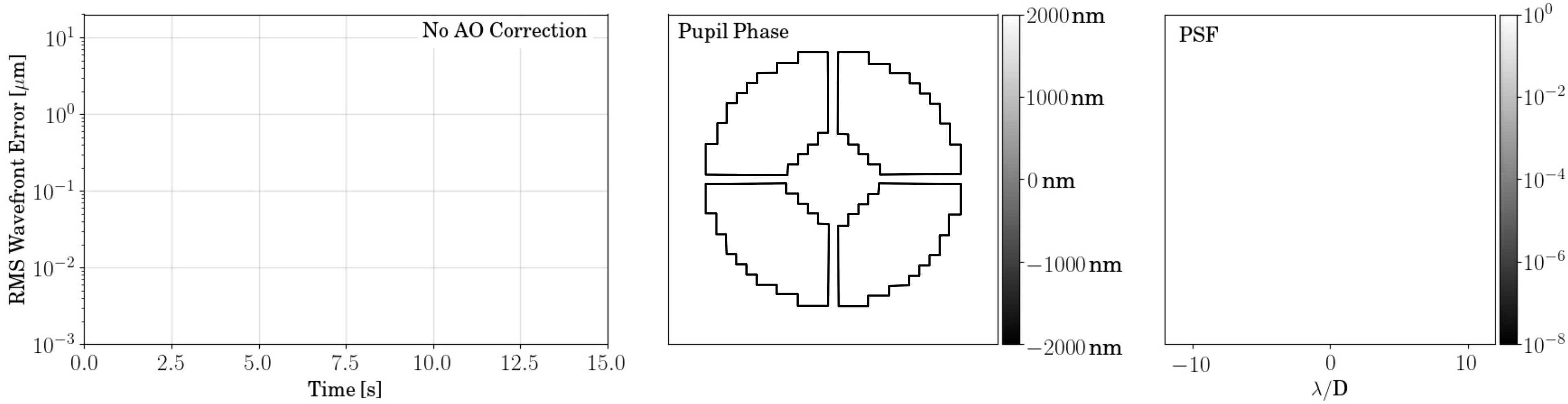
Guyon & Males 2017: use SVD to solve for the filter in terms of the training set

$$min_{\boldsymbol{F}^i} < ||\boldsymbol{F}^i \boldsymbol{h}(t) - \boldsymbol{\omega}_i(t + \delta t)||^2 >_t$$

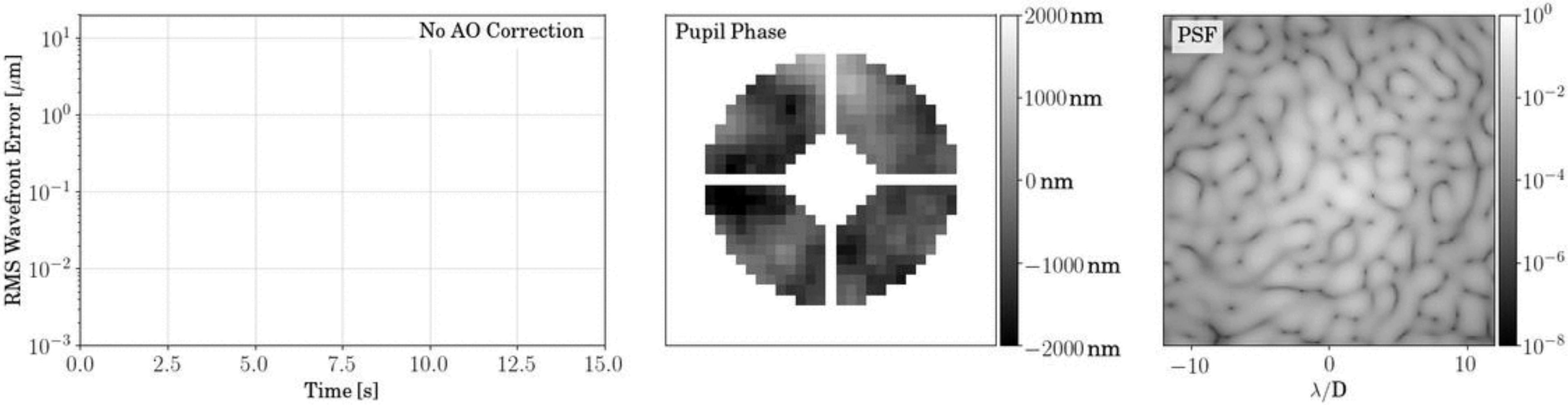
$$min_{\boldsymbol{F}^{\boldsymbol{i}}} ||\boldsymbol{F}^{\boldsymbol{i}} \boldsymbol{D} - \boldsymbol{P}_{\boldsymbol{i}}||^2$$

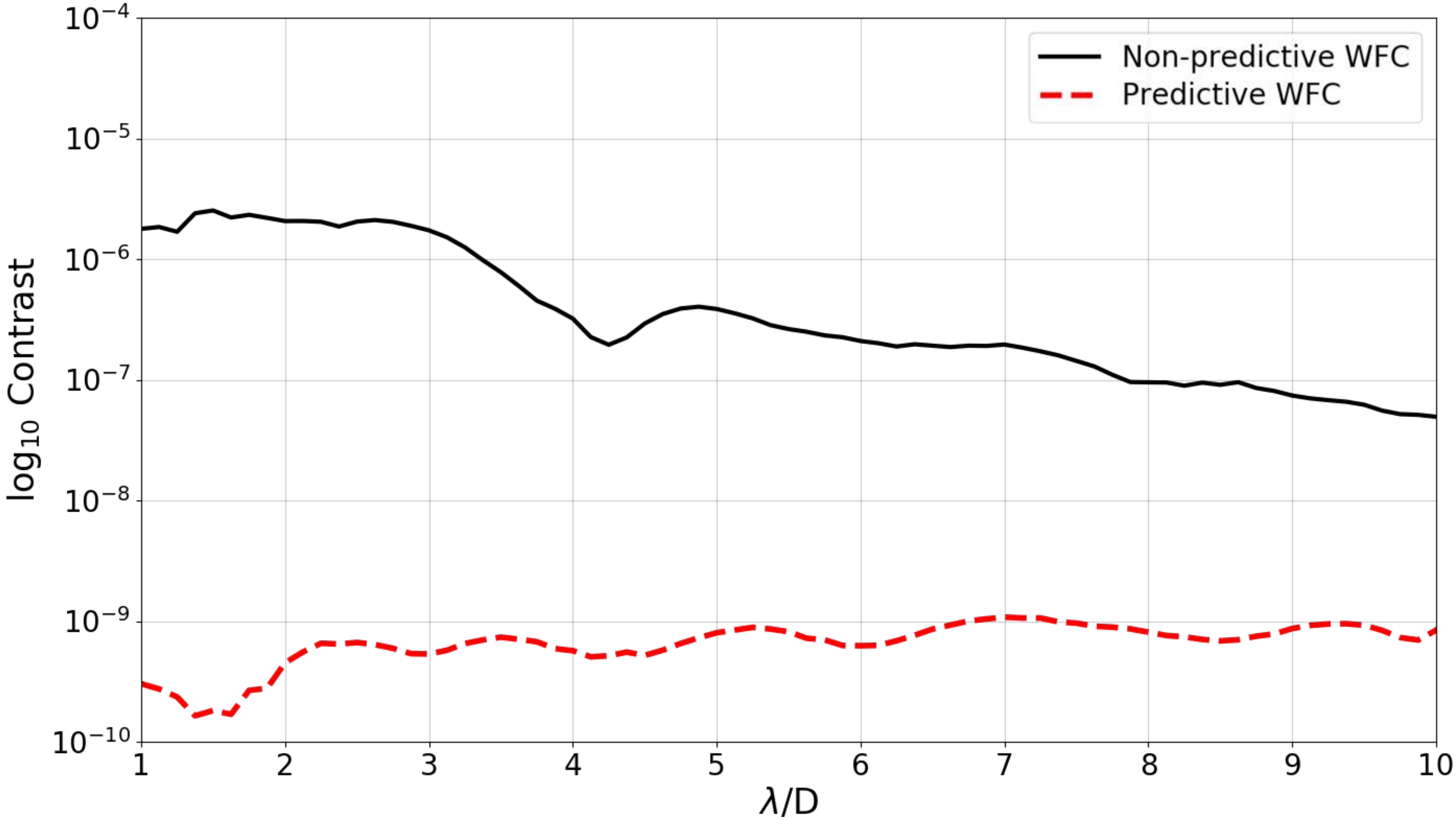
$$\boldsymbol{F}^i = \left(\left(\boldsymbol{D}^T \right)^+ \tilde{\boldsymbol{P}}_i^T \right)^T$$

Guyon & Males 2017: how well does it work in a simple frozen flow simulation?

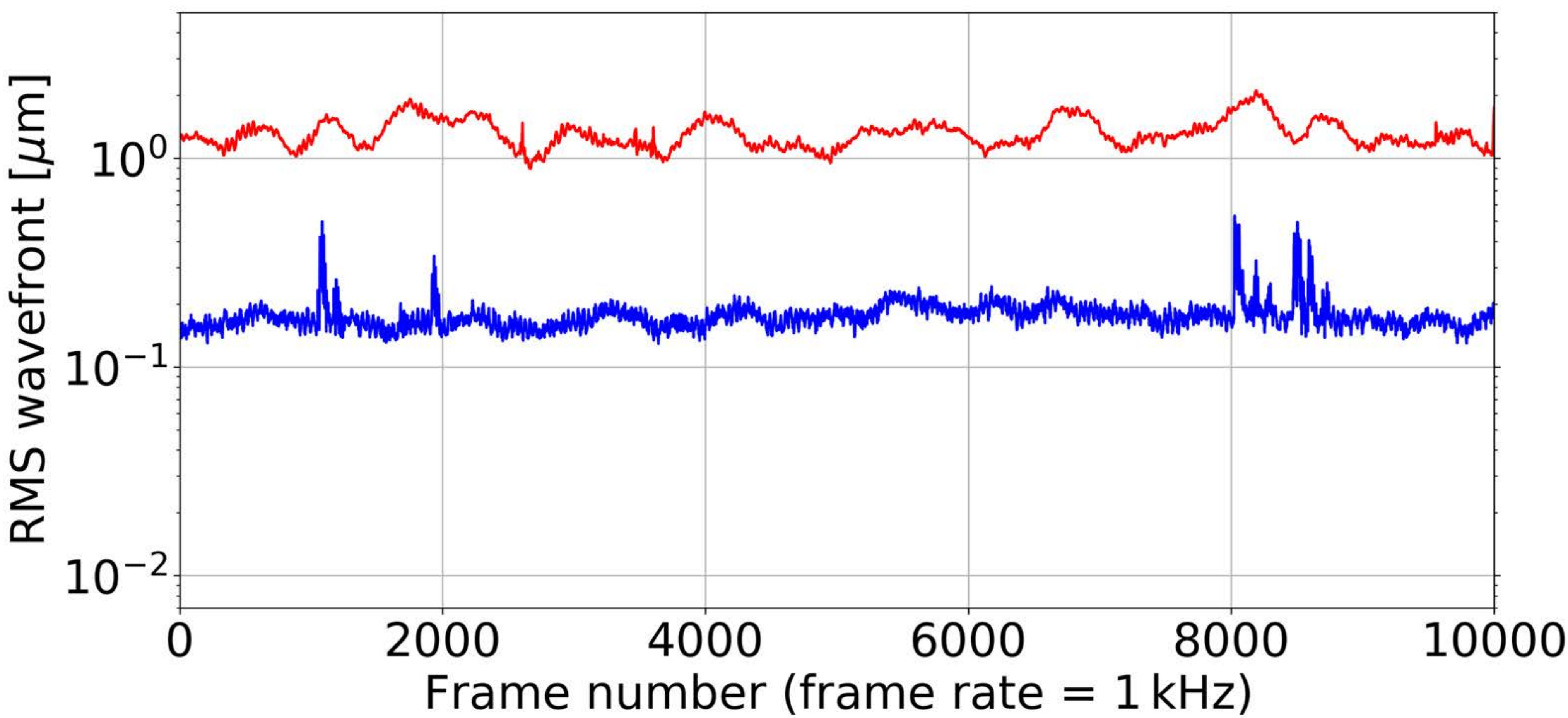


Guyon & Males 2017: how well does it work in a simple frozen flow simulation?





Non-predictive WFC
Predictive WFC
$\log_{10}$ Contrast
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
1
2
3
4
5
6
7
8
9
10
λ/D



RMS wavefront [μm]
10^0
10^−1
10^−2
0
2000
4000
6000
8000
10000
Frame number (frame rate = 1 kHz)

Summary of Wavefront Reconstruction

- Because most wavefront sensors don't measure the phase directly, we must "reconstruct" the phase based on our information from the wavefront sensor
- Usually, WF reconstruction is done with a matrix
  - We can reconstruct the wavefront "zonally" in DM actuator space or "modally" e.g. in Zernike modes
  - We typically measure the interaction matrix experimentally
  - We compute the pseudo-inverse of the interaction matrix to find the reconstruction matrix
- Predictive WFC is motivated by the time delay between reconstructing the phase and correcting the WF with the DM (often a few milliseconds or less)